

**Applications of Perron-Frobenius  
Theory  
to Population Dynamics**

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# POPULATION DEMOGRAPHY

Demography is the study of  
population in terms of its growth and  
decay, fertility and mortality, ....

John Impagliazzo, 1984

Models:

Continuous

Alfred J. Lotka [1880 - ]

Discrete and Linear

P.H. Leslie, *Econometrica*, (1945, 1948)

E.G. Lewis (1941)

H. Bernadelli (1941)

Single sex

(St)age distribution

## Age distribution - Leslie Matrix transition (mortality) matrix,

$t_j$  prob of survival from  $j$  to  $j + 1$

$$0 \leq t_j \leq 1$$

$$T = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ t_1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & t_{n-1} & 0 \end{pmatrix}$$

the fertility matrix

$f_j$  no of exp newbrns indiv age  $j$

$$0 \leq f_j$$

$$F = \begin{pmatrix} f_1 & \cdots & f_{n-1} & f_n \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

projection matrix.

$$P = T + F = \begin{pmatrix} f_1 & \cdots & f_{n-1} & f_n \\ t_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{n-1} & 0 \end{pmatrix}$$

$$Px = \begin{pmatrix} f_1x_1 + \cdots + f_nx_n \\ t_1x_1 \\ \vdots \\ t_{n-1}x_{n-1} \end{pmatrix}$$

Standard model of population demography:

$$x^0 \geq 0$$

$$x^k = Px^{k-1}, \quad k = 1, 2, \dots$$

“stable solution”

$$Px = rx, \quad x \geq 0, r > 0$$

$$\det(\lambda I - P) = \lambda^n - (c_1 f_1 \lambda^{n-1} + \dots + c_n f_n)$$

$$c_1 = 1, \quad c_i = t_1 t_2 \dots t_{i-1} \geq 0$$

survival prob from age 1 to age  $i + 1$

$$1 = c_1 f_1 / \lambda^{-1} + \dots + c_n f_n / \lambda^{-n} =: p(\lambda)$$

$p(\lambda)$  is mon decr from  $\infty$  to 0

has unique positive root  $r$

all other roots  $\leq$  in modulus

$$1 = c_1 f_1 / r \dots + c_n f_n r$$

corresp eigenvector

$$x = (c_1, c_2 / \lambda, \dots, c_n \lambda^{n-1})^t$$

$$p(1) = c_1 f_1 + \dots + c_n f_n =: R_0$$

no. newborns from one indiv. in  
lifetime

net reproductive rate

$$R_0 \geq 1 \iff r \geq 1$$

## GENERAL CASE

$$T \geq 0, \quad \sum_j t_{ij} \leq 1$$

$t_{ij}$  prob trans  $(j) \rightarrow (i)$

$$\lim_k T^k x = 0, \forall x \iff \rho(T) < 1$$

$$F \geq 0$$

$f_{ij}$  av. new by  $(j)$  in  $(i)$

$$P = T + F$$

## VERY IMPORTANT MODEL

Population = All mathematicians

Classified by no. of papers in year  $k$

Death = leaving the profession

$t_{ij}$  prob that math with  $j$  papers in year  $k$  has  $i$  papers in year  $k + 1$

$f_{ij}$  prob that in year  $k$  math with  $j$  papers produces Ph.D. with  $i$  papers in year  $k + 1$

# PERRON-FROBENIUS THEORY

*P* irreducible:

not in form by permutation similarity

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

$A_{11}$  and  $A_{22}$  nontrivial

*P* primitive :  $P^k > 0$

*P-F Theorem for irreducible matrices*

Let  $P$  be an irreducible nonnegative matrix. Then

- (a) The spectral radius  $\rho(P)$  of  $P$  is positive and it is an algebraically simple eigenvalue of  $P$  (the *Perron root*) with corresponding left and right positive eigenvectors (the *Perron vectors*), which are unique up to scalar multiples.
- (b) The spectral radius of  $P$  is the unique eigenvalue with a nonnegative eigenvector.
- (c) The spectral radius of the matrix  $P$  increases (strictly), resp. decreases, if any entry of it increases, resp. decreases.

*Fundamental Theorem of Demography:*

Let  $P$  be the projection matrix of a standard population model  $x^k = P^k x^0$ ,  $k = 0, 1, \dots$ . Suppose that  $P$  is primitive with spectral radius  $\rho(P) = r$  and has left and right Perron vectors  $v^t$  and  $u$  resp. normalized so that  $v^t u = 1$ . Then  $(\lim_{k \rightarrow \infty} (P/r)^k = uv^t)$

$$\lim_{k \rightarrow \infty} x^0 / r^k = (v^t x^0) u.$$

Consequently, if  $|x^k|$  denotes the total population at time  $k$  then

$$\lim_{k \rightarrow \infty} |x^k| = \begin{cases} 0 & \text{if } r < 1, \\ |(v^t x^0) u| & \text{if } r = 1, \\ \infty & \text{if } r > 1. \end{cases}$$

# NET REPRODUCTIVE RATE $R_0$

$$\rho(T) < 1$$

$$I + T + T^2 + \dots = (I - T)^{-1}$$

dist of newborns from init dist over  
lifetime

$$Fx + FTx + FT^2x + \dots = F(I - T)^{-1}x$$

$$Q := F(I - T)^{-1}$$

Next Generation matrix

$$R_0 := \rho(Q)$$

Model:

$q_{ij}$  lifetime Ph.D's born with  $i$  papers  
to math born with  $j$  papers

*Stability and comparison theorem:* Assume that a projection matrix  $P = T + F$  is irreducible with  $T$  and  $F$  nonzero. Denote the growth rate  $\rho(P)$  by  $r$  and the net reproductive rate  $\rho(Q)$  by  $R_0$ . Then

$$R_0 > 0,$$

$$\rho(T + F/R_0) = 1,$$

and one of the following holds:

$$1 < r < R_0,$$

$$r = 1 = R_0,$$

$$0 < R_0 < r < 1.$$

cf. Stein-Rosenberg

## SOME PROOFS

$R_0 > 0$  follows from

**Proposition:** Let  $T$  and  $F$  be nonnegative matrices with  $\rho(T) < 1$  and  $F \neq 0$ . Suppose  $T + F$  is irreducible and  $Q = F(I - T)^{-1}$ . Then, after a permutation similarity,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & 0 \end{pmatrix},$$

where  $Q_{11}$  is a nontrivial irreducible nonnegative matrix,  $Q_{12}$  is a nonnegative matrix every column of which has a positive entry, and the 0 rows of  $Q$  correspond to the 0 rows of  $F$ , if any.

S (1984), Szyld (1985)

$$y^t \not\geq 0, \quad R_0 y^t = y^t Q = y^t F (I - T)^{-1}$$

$$y^t (T + F/R_0) = y^t$$

Hence

$$\rho(T + F/R_0) = 1$$

Case  $R_0 > 1$

$$T + F/R_0 \not\leq T + F \not\leq R_0 T + F$$

Hence

$$1 = \rho(T + F/R_0) < \rho(T + F) = r$$

$$r < \rho(R_0 T + F) = R_0$$

## INTUITIVE CONTENT

Suppose

$$r = \rho(P) = \rho(T + F) > 1$$

$$P^r x_0 \rightarrow \infty$$

**Aim:** Stationary Population

Birth/death control

$$T + F \longrightarrow (T + F)/r$$

Birth control  $\quad \uparrow \downarrow$

$$T + F \longrightarrow T + F/R_0$$

**Intuitive:** Second more radical

$$R_0 > r$$

## A GENERALIZATION

Achieving a given growth rate  $s$  by  
scaling  $F$

***THEOREM:*** Let  $P, T$  and  $F$  satisfy the previous conditions. For  $s > \rho(T)$  define

$$q(s) = \rho(F(I - T/s)^{-1})/s.$$

Then  $q(s) > 0$ . Let  $P(s) = T + F/q(s)$ . Then its growth rate,  $\rho(P(s))$ , is  $s$ , and its net reproductive rate is

$$R_0(s) = R_0/q(s).$$

Further, one of the following holds:

$$\begin{aligned} 1 &= s = R_0(s), \\ 1 &< s < R_0(s), \\ 0 &< R_0(s) < s < 1. \end{aligned}$$

**Previous Theorem:**  $s = r, q(s) = 1$